Vibration isolation basics

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These set of slides were originally written for the PROSPECT collaboration who needed to transport the detector from New Haven to Knoxville. They are certainly of general interest to anyone who wants to get the basic understanding behind vibration analysis

Problem

- We need to transport a 10 ton object of dimension 2 x 2 x 2 meters over 1000 km.
- A typical truck has mass of 14 tons.
- Object will be placed on the truck bed on top of some shipping platform.
- What are the basic considerations regarding vibration ?
- Problem of vibrations and isolation is quite common in many large and small systems. We will focus on the above situation just to allow a mental picture, but the formalism to be developed is applicable in many places.

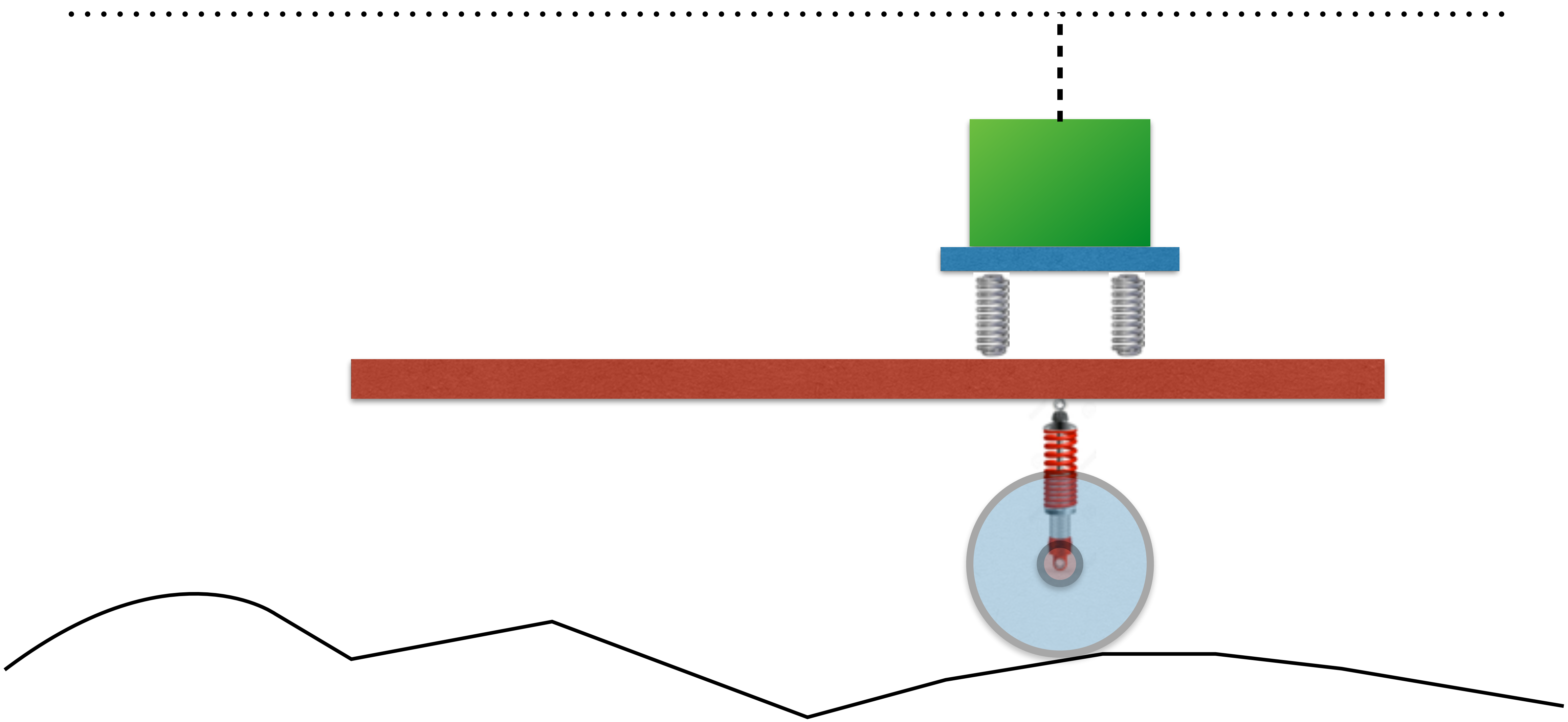
Reference material

- ***Any advanced text book on mechanics will have the topic.***
- ***Wikipedia has a nice description, but does not have the formalism : https://en.wikipedia.org/wiki/Vibration_isolation***
- ***Many texts will cover the mathematics, but keep it too general.***
- ***The best resource might be MIT lectures by Prof. Vandiver and Gossard.***
- ***https://ocw.mit.edu/courses/mechanical-engineering/2-003sc-engineering-dynamics-fall-2011/***
- ***We will briefly touch on modal analysis here, but my plan is to write a set of slides on this later.***

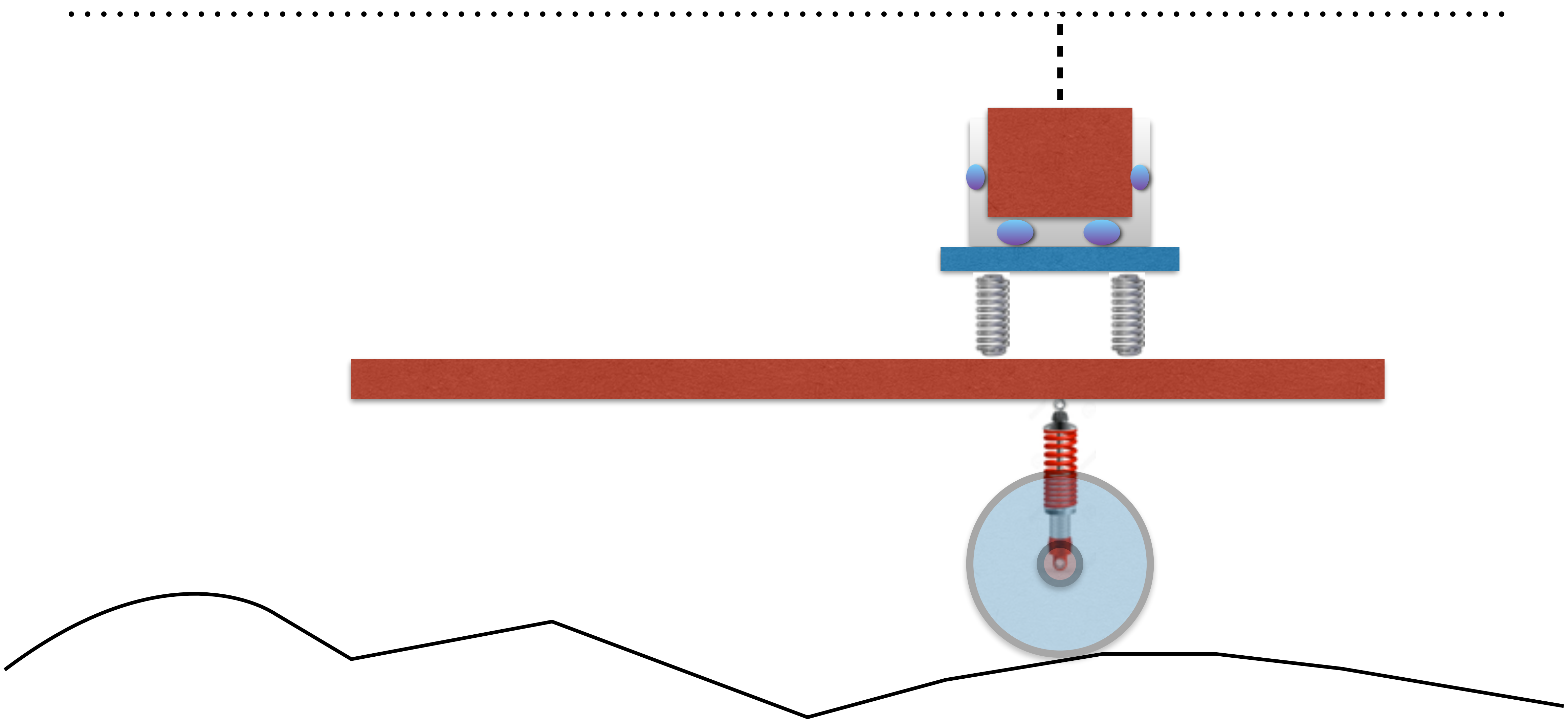
Some assumptions

- ***We will assume that the isolation of the overall object is needed, and we ignore isolation of any components inside the object. In fact, the same analysis can be performed for each element of the object.***
- ***Powerful software will perform a multidimensional analysis and identify various modes of vibration. Our goal here is to provide just enough understanding so that we can follow more sophisticated analysis.***
- ***The object will be placed on an isolation platform on the truck.***
- ***We assume a linear system. This means that if the truck shakes at a certain frequency then the object and all its parts will vibrate at the same frequency. this is the basis of Fourier analysis.***
- ***There are two issues:***
    - ***What is the steady state response of the detector ? (we work on this first).***
    - ***What is the response to a shock event ?***

First we make a visual picture

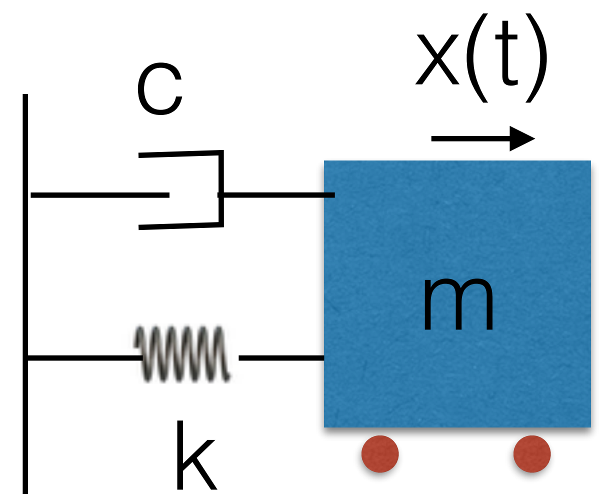


It is good to imagine that the load is attached to an imaginary overhead line as it moves along the road to understand the impact of the suspension system.



You can extend the imagination of isolation to inside of the object. The inside object continues on an imaginary line while everything around it is vibrating.

Simple 1 DOF system



$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0$$

Set $x(t)=Xe^{i\omega t}$

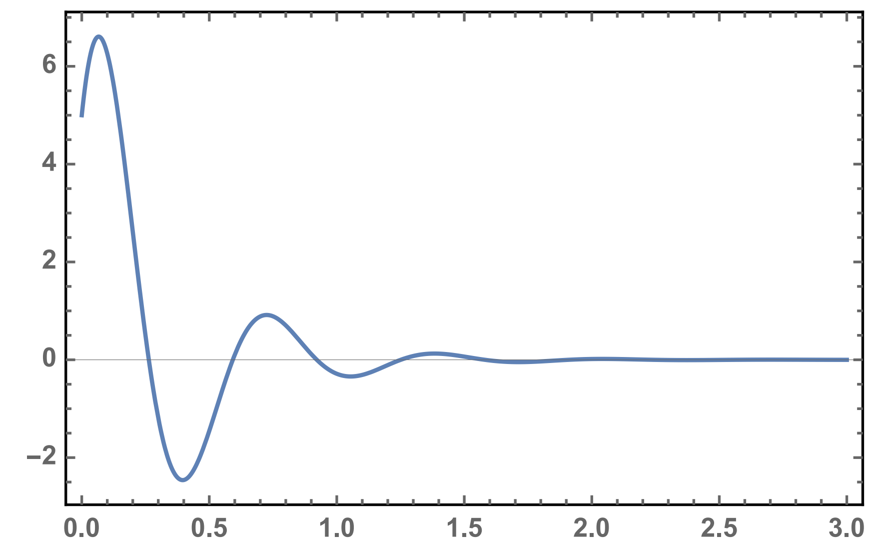
$-\omega^2 mX+i\omega cX+kX=0$ This yields a solution for ω

$$\omega=-i\frac{c}{2m}\pm\sqrt{\frac{k}{m}}\sqrt{1-\frac{c^2}{4mk}}$$

Set $\xi=\frac{c}{2m\omega_n}$, $\omega_n^2=\frac{k}{m}$ is the natural frequency

$\omega=-i\xi\omega_n\pm\omega_n\sqrt{1-\xi^2}$, the complete solution is then

$$x(t)=e^{-\xi\omega_n t}\left(x_0\,\mathrm{Cos}(\omega_d t)+\frac{v_0+\xi\omega_n x_0}{\omega_d}\mathrm{Sin}(\omega_d t)\right)$$

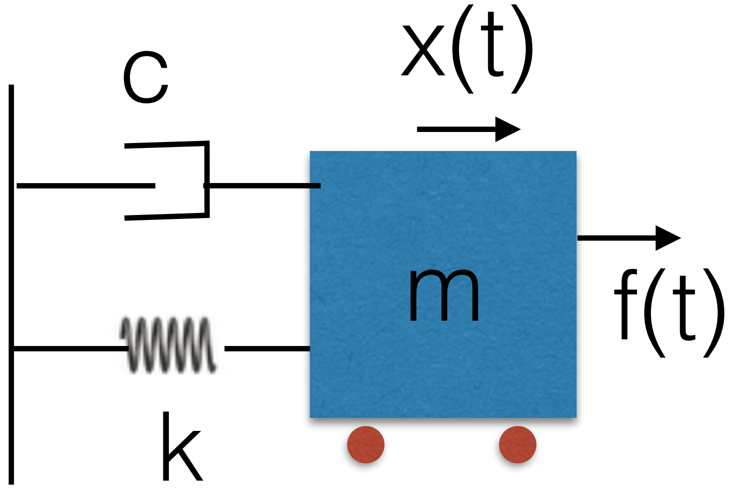


where $\omega_d=\omega_n\sqrt{1-\xi^2}$ is called the damped frequency.

If we assume small damping, then the intercept of this motion is the initial displacement x_0 and the initial slope corresponds to $\sim v_0$

This is a well known result that everyone should know

Simple forced system



$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=f(t)$$

Fourier: $x(t)\Leftrightarrow X(\omega);\ f(t)\Leftrightarrow F(\omega)$

$$-\omega^2 mX+i\omega cX+kX=F(\omega)$$

$$\frac{X}{F}=\frac{1}{k}\left[\frac{1}{(1-\omega^2/\omega_n^2)+2i(\omega/\omega_n)\xi}\right]$$

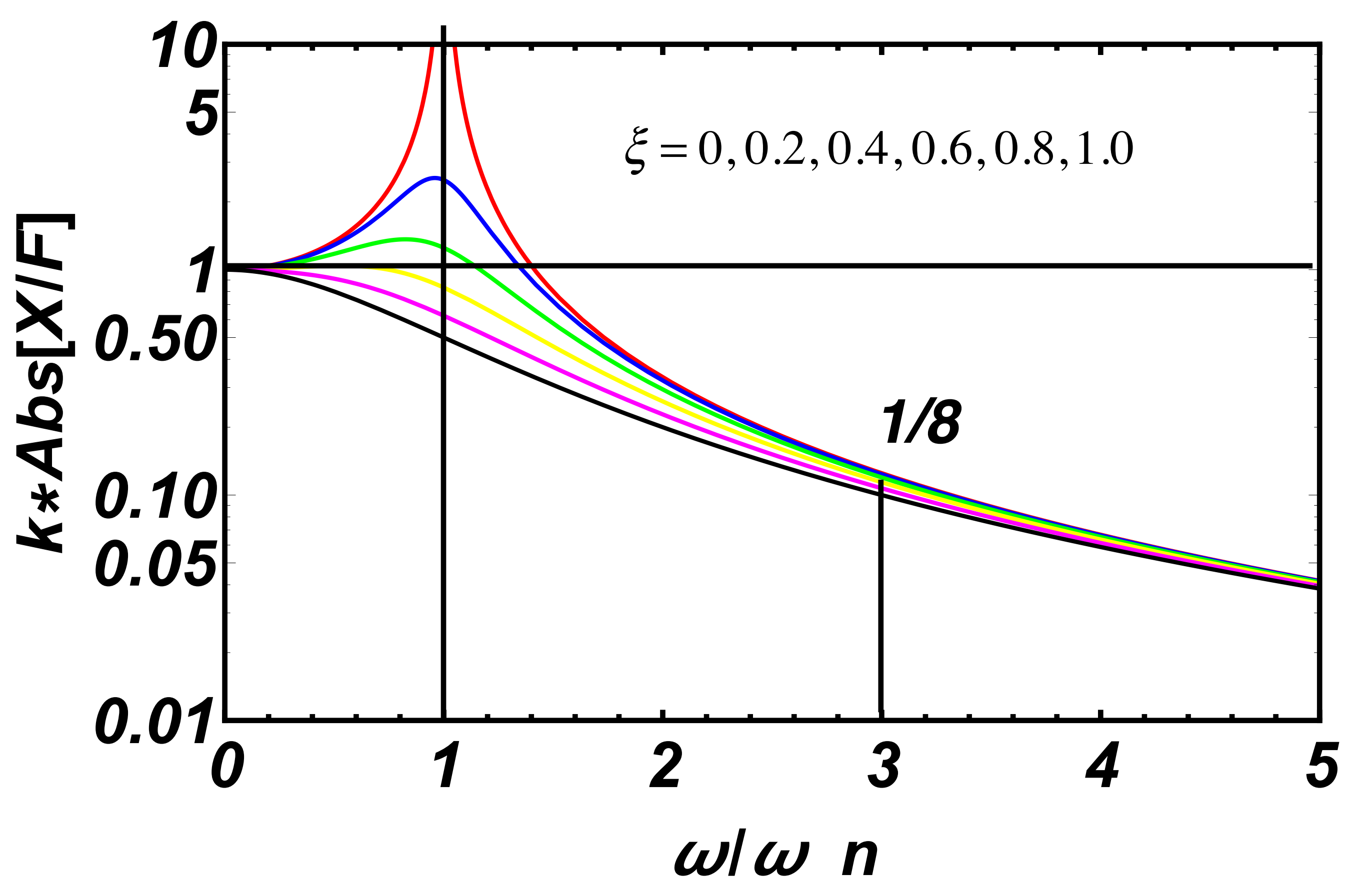
Units k: N/m, c: N/m/s
f(t): applied force in N

This is called a transfer function

$$\omega_n^2=\frac{k}{m}\qquad \xi=\frac{c}{2\omega_n m}$$

$$\text{Natural Freq}=\nu_n=\frac{\omega_n}{2\pi}\quad \text{Hz}$$

Damping $=\xi$ is unitless



Comments

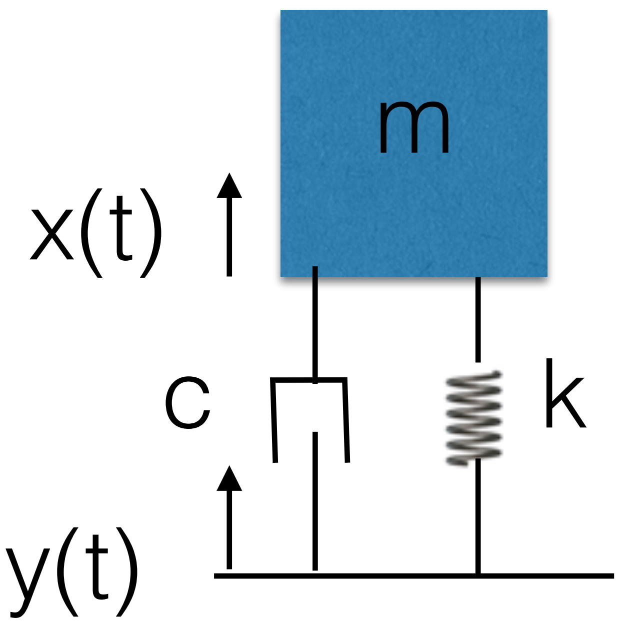
- ***Best way to reduce response is to detune the natural frequency by a factor of ~3 compared to the driving frequency.***

- ***Factor of 3 detuning gets you factor 1/8 in response.***

- ***Driving excitation in the steady state may have a wide spectrum.***

- ***Damping is needed in case of impulse to stop long term vibration at natural frequency.***

Floor vibration



At equilibrium spring has to support force of mg

$k\delta = mg$ where δ is the motion after loading with mg

$$k = \frac{mg}{\delta} \quad \Rightarrow \omega_n^2 = \frac{k}{m} \quad \Rightarrow \omega_n = \sqrt{\frac{g}{\delta}}$$

Assume $x(t)$ is the motion after reaching equilibrium.

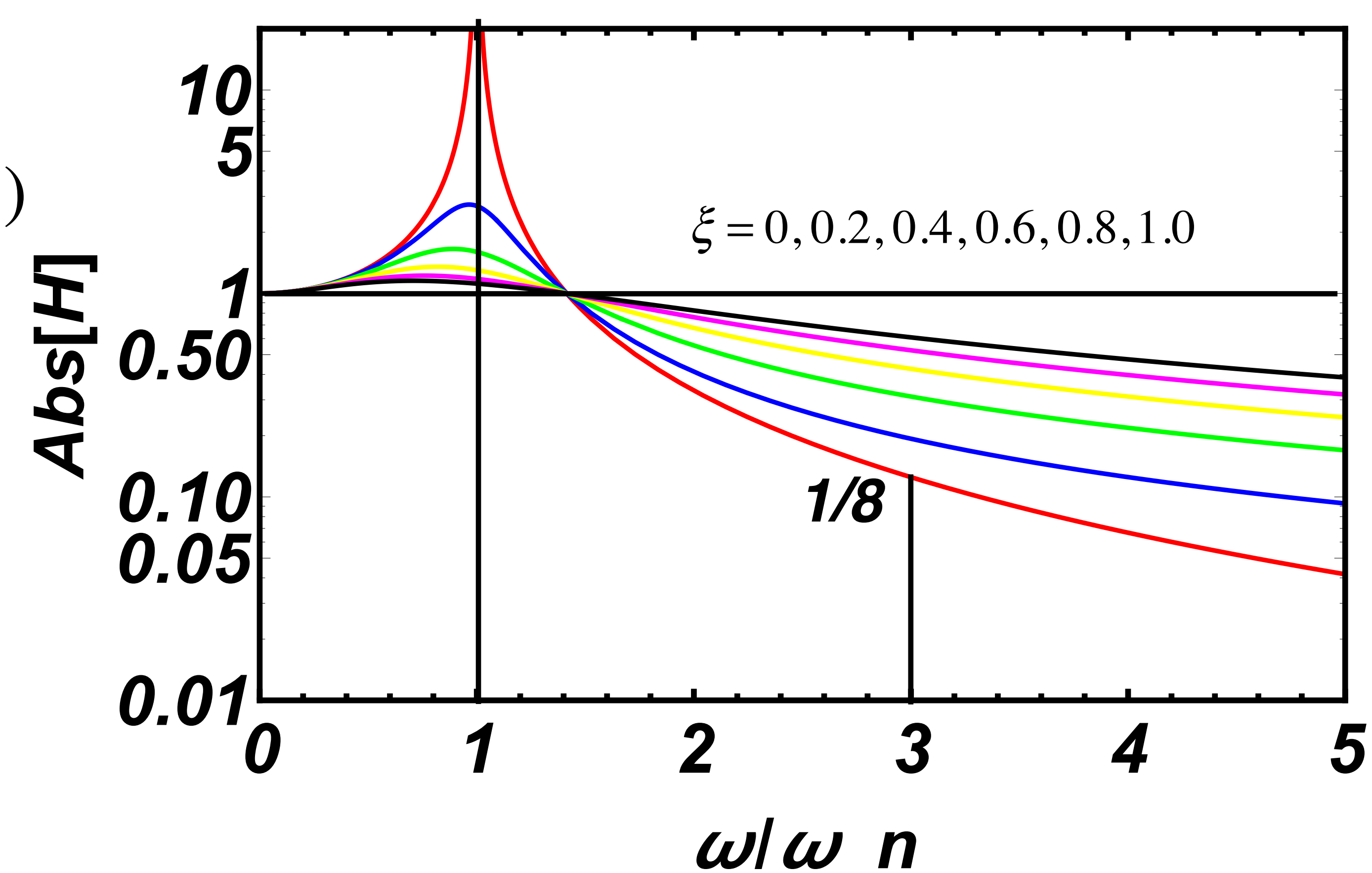
Actual frequency at maximum shifts with damping.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = c\frac{dy}{dt} + ky$$

Fourier: $x(t) \Leftrightarrow X(\omega) \quad y(t) \Leftrightarrow Y(\omega)$

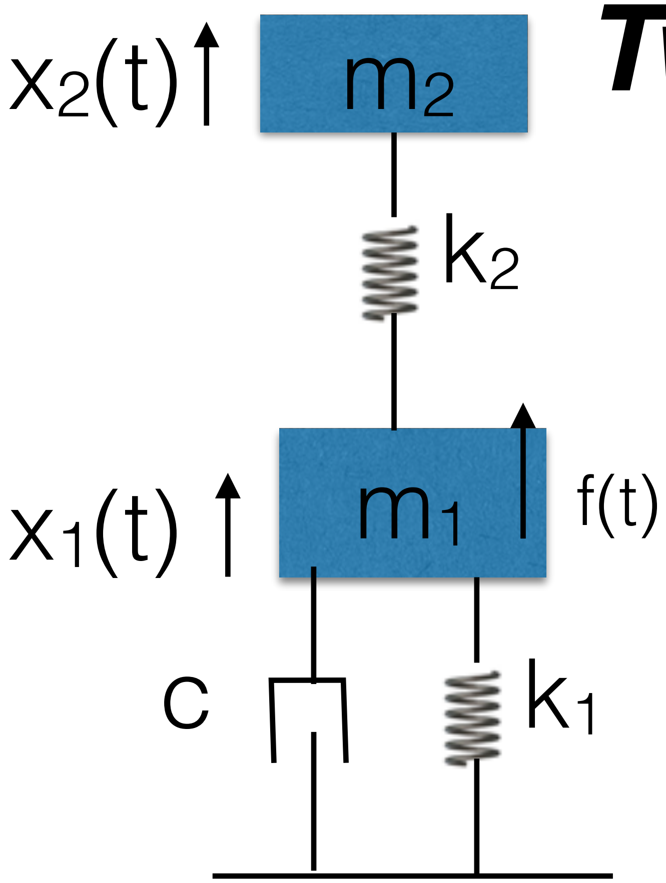
$$-\omega^2 mX + i\omega cX + kX = i\omega cY + kY$$

$$H(\omega) = \frac{X}{Y} = \frac{1 + 2i\frac{\omega}{\omega_n}\xi}{1 - \frac{\omega^2}{\omega_n^2} + 2i\frac{\omega}{\omega_n}\xi}$$



Notice an important difference at larger values. Damping does not help

Two degrees of freedom with f(t)



$$m_1 \frac{d^2 x_1}{dt^2} + c\frac{dx_1}{dt} + k_1 x_1 + k_2(x_1 - x_2) = f(t)$$

$$m_2 \frac{d^2 x_2}{dt^2} + k_2 x_2 - k_2 x_1 = 0$$

In Fourier space

$$\begin{pmatrix} -\omega^2 m_1 + i\omega c + (k_1 + k_2) & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} F(\omega) \\ 0 \end{pmatrix}$$

First assume $c = 0$ to get the normal modes

solving for ω gives

$$\omega^2 = \frac{1}{2}\left[\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \pm \frac{(-4k_1 k_2 m_1 m_2 + (k_2 m_1 + k_1 m_2 + k_2 m_2)^2)^{1/2}}{m_1 m_2}\right]$$

Ratio of amplitudes for the two modes are

$$\frac{X_2}{X_1} = \frac{-\omega^2 m_1 + k_1 + k_2}{-k_2}$$

Here one plugs in the two eigenvalues of ω_+ and ω_-

We also define $\omega_1^2 = \dfrac{k_1}{m_1}$; $\omega_2^2 = \dfrac{k_2}{m_2}$

Calculate these for special cases

Some parameters

- *Mass of truck trailer m_1 =15000 kg*
- *Mass of object+platform m_2=10000 kg*
- *Assume that the weight of the load compresses the truck spring by δ;*
- *For example, $\delta \sim 0.1$ m for truck then $k_1 = m_1 g/\delta \sim 1.5 \times 10^6$ N/m*
- *Truck suspension could be much stiffer $k_1 = 1.5\times 10^7$ N/m (frequency = Sqrt(k_1/m_1)/2Pi = 1.59 Hz — 5.0 Hz). For modern trucks this can be adjustable.*
- *If $\delta \sim 0.05$ m for just the detector on the truck then $k_2 = m_2 g/\delta \sim 2.0 \times 10^6$ N/m*
- *Examine $c/(2\omega_1 m_1)$ ranging from 0 to 10.*
- *Typical values $c \sim 3\times 10^5 - 3\times 10^6$ N/(m/s).*
- *We will now use these to make a table for the normal modes for this system.*

Normal modes

m_1	m_2	k_1	k_2	$\omega_1/2\pi$	$\omega_2/2\pi$	$\omega_+/2\pi$	$\omega_-/2\pi$	$(X_2/X_1)_+$	$(X_2/X_1)_-$
1	1	1	1	1	1	$(1+\sqrt{5})/2$ $=1.618$	$(\sqrt{5}-1)/2$ $=0.618$	$(\sqrt{5}-1)/2$ $=0.618$	$(-\sqrt{5}-1)/2$ $=-1.618$
15000	10000	$1.5*10^6$	$2*10^6$	1.59	2.25	3.10	1.15	1.11	-1.36
15000	10000	$1.5*10^6$	$21*10^6$	1.59	7.29	9.47	1.22	1.45	-1.03
15000	10000	$15*10^6$	$21*10^6$	5.03	7.29	10.03	3.66	1.12	-1.34

All are in SI units. kg, N, m, sec, Hz etc.

First row is a well known result from text books.

Transfer Function

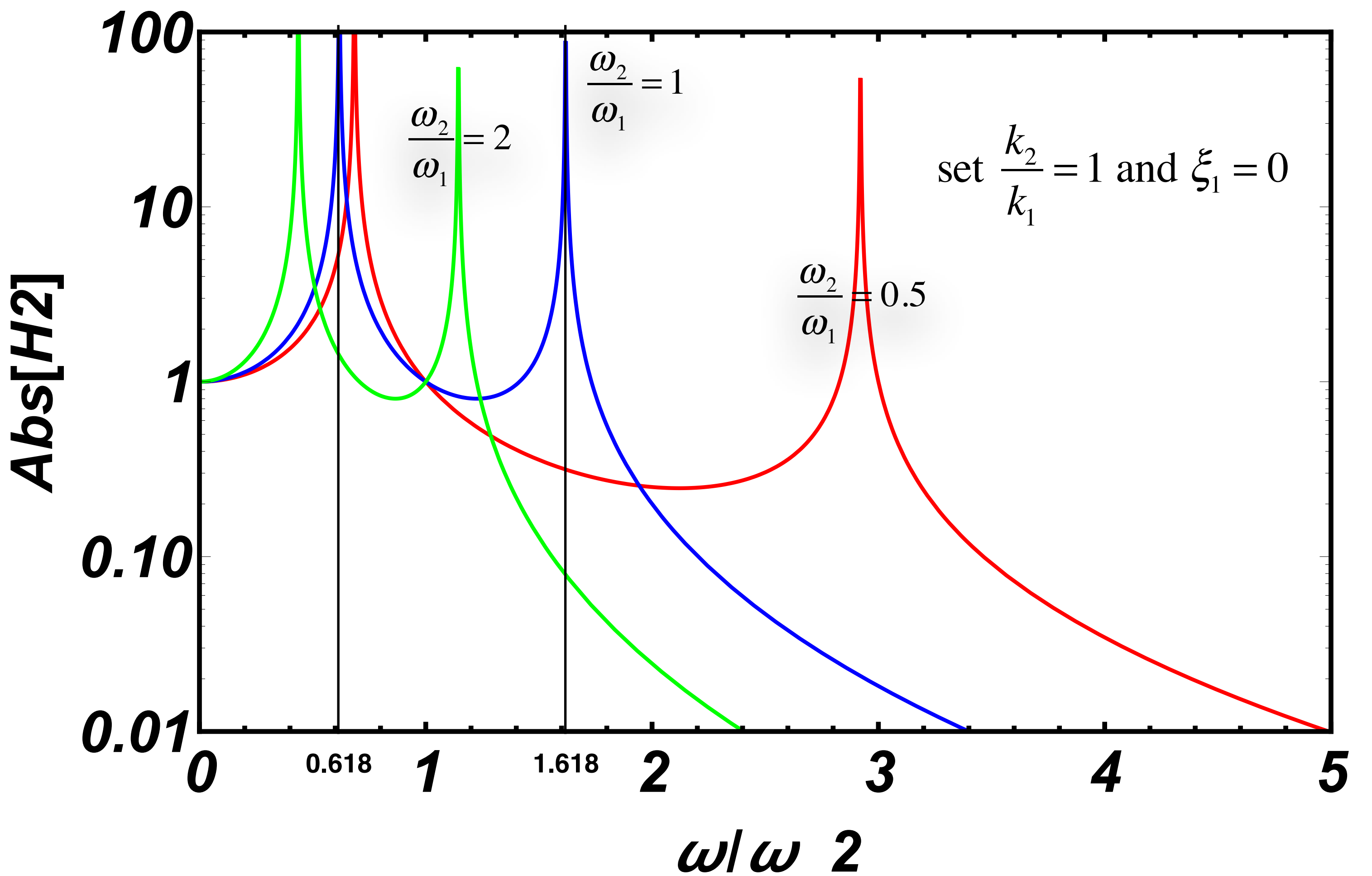
$$\frac{X_1}{F} = \frac{-(k_2 - m_2\omega^2)}{k_2^2 - (k_1 + k_2 - m_1\omega^2 + ic\omega)(k_2 - m_2\omega^2)}$$

$$\frac{X_2}{F} = \frac{k_2}{k_1k_2 + ic\omega(k_2 - m_2\omega^2) - (k_2m_1 + k_1m_2 + k_2m_2)\omega^2 + m_1m_2\omega^4}$$

$$H_2(\omega) = \frac{X_2}{F/k_1} = \frac{1}{1 + i2\frac{\omega_2}{\omega_1}\frac{\omega}{\omega_2}\xi_1\left[1 - \frac{\omega^2}{\omega_2^2}\right] - \left[1 + \frac{k_2}{k_1} + \frac{\omega_2^2}{\omega_1^2}\right]\frac{\omega^2}{\omega_2^2} + \frac{\omega_2^2}{\omega_1^2}\frac{\omega^4}{\omega_2^4}}$$

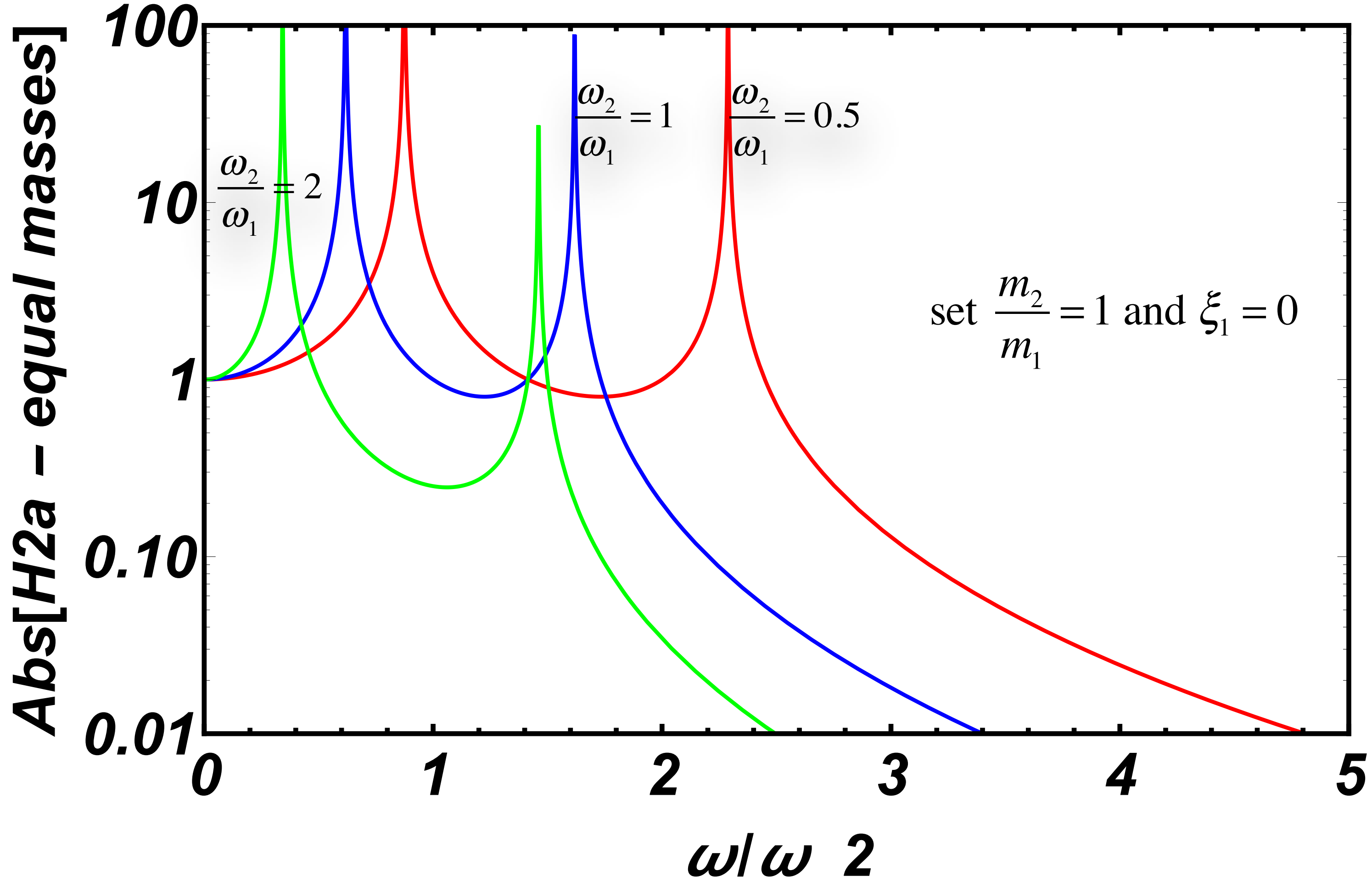
$$\omega_1^2 = \frac{k_1}{m_1} \qquad \xi_1 = \frac{c}{2\omega_1 m_1}$$

Damping = ξ_1 is unitless



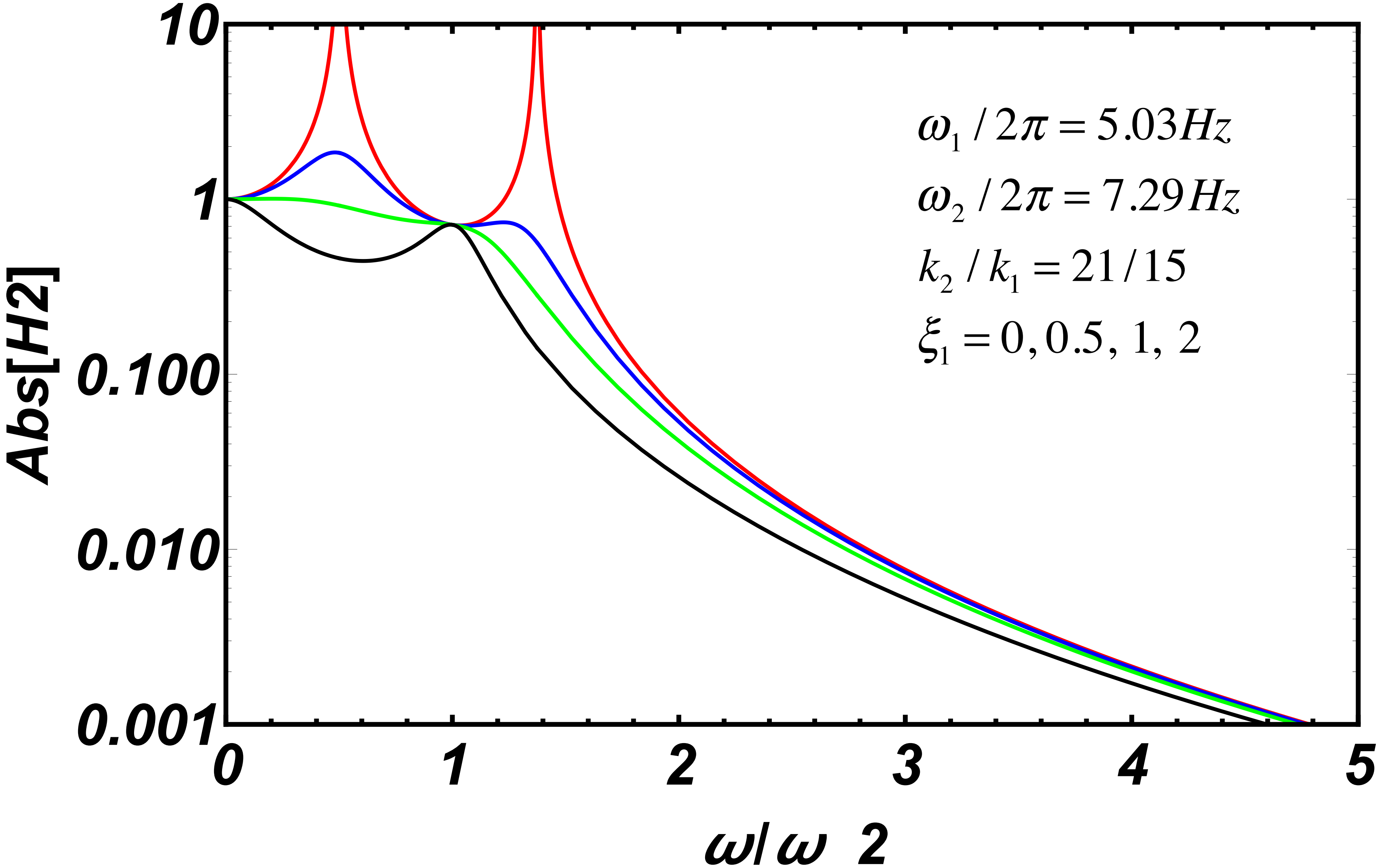
It is better to have $\omega_2 > \omega_1$

Transfer function with no damping

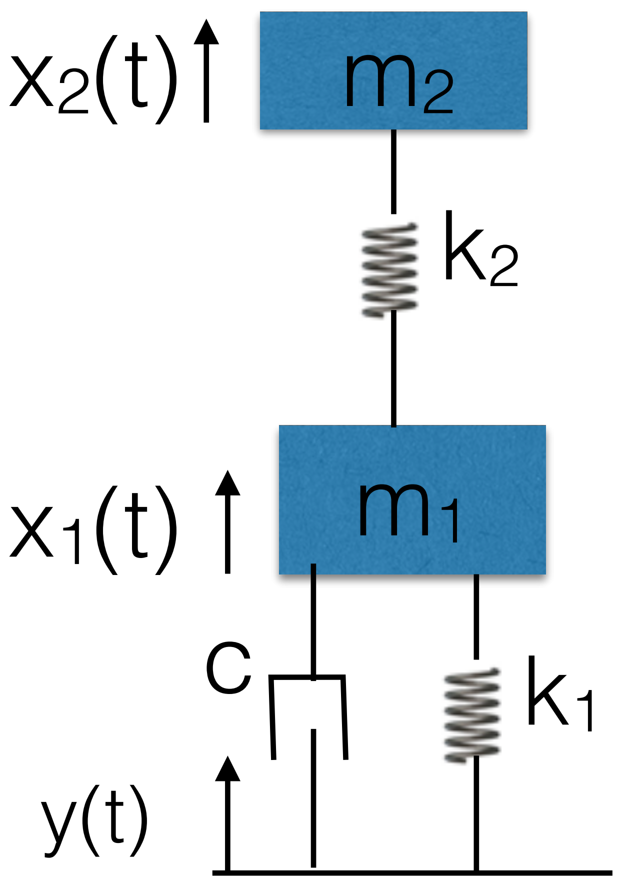


- ***This is the same plot, but now the masses are kept equal and the stiffness is changed to change the frequency ratio.***
- ***This is saying that it is better to have the frequency of the shipping platform higher (or stiffer) than the truck suspension.***
- ***k2/m2 > k1/m1 will reduce the motion of the overall detector with respect to the road. But this will lower one of the frequencies to very low values causing big motions of the truck bed.***

Transfer function with damping



There is considerable more detuning with 2 degrees of freedom. Question: what are parameters of a truck trailer ?



Two degrees of freedom with floor vibration

$$m_1 \frac{d^2 x_1}{dt^2} + c\frac{dx_1}{dt} + k_1 x_1 + k_2(x_1 - x_2) = c\frac{dy}{dt} + k_1 y$$

$$m_2 \frac{d^2 x_2}{dt^2} + k_2 x_2 - k_2 x_1 = 0$$

In Fourier space

$$\begin{pmatrix} -\omega^2 m_1 + i\omega c + (k_1 + k_2) & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} (ic\omega + k_1)Y(\omega) \\ 0 \end{pmatrix}$$

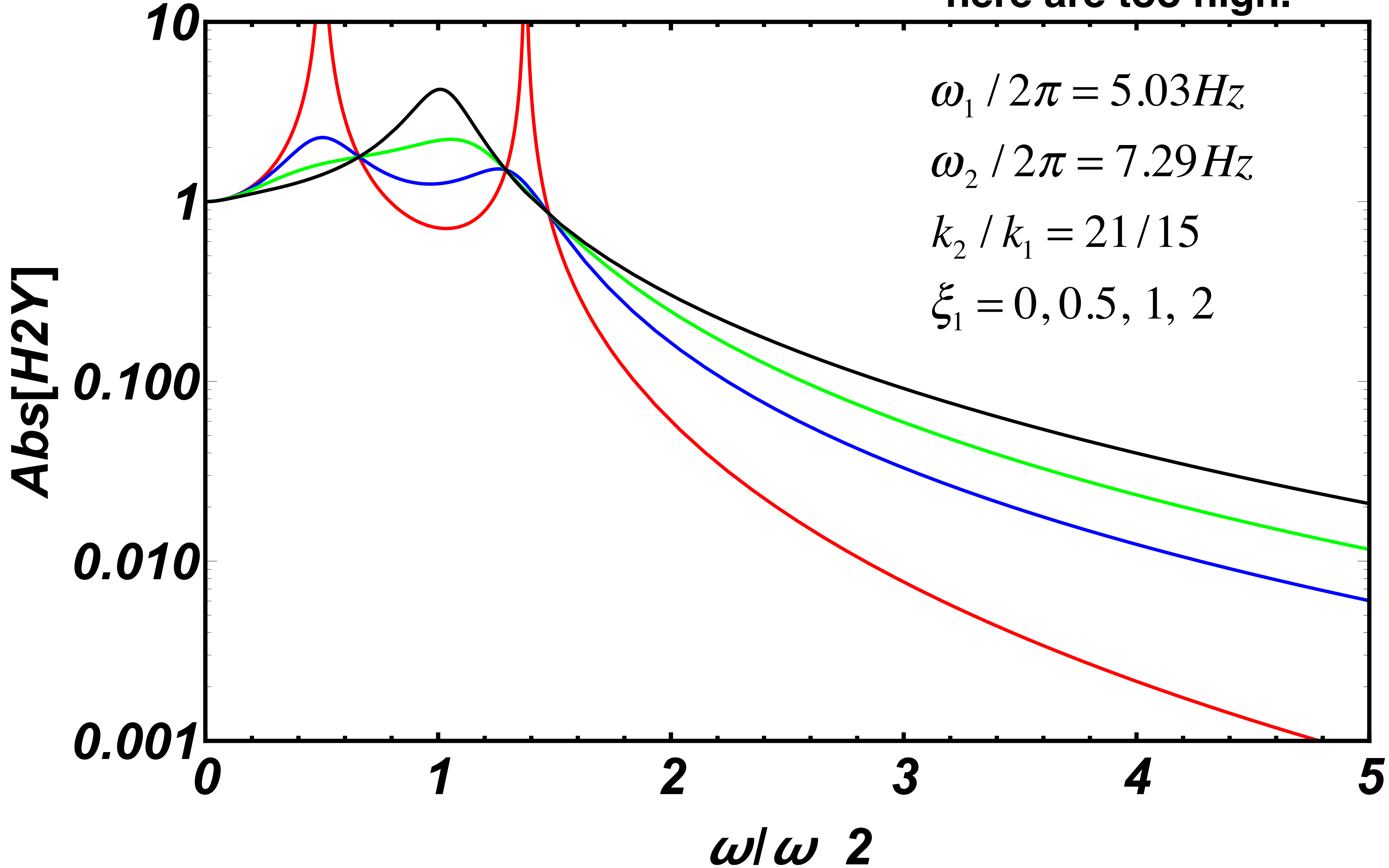
$$\frac{X_1}{Y} = \frac{-(k_2 - m_2\omega^2)(ic\omega + k_1)}{k_2^2 - (k_1 + k_2 - m_1\omega^2 + ic\omega)(k_2 - m_2\omega^2)}$$

$$\frac{X_2}{Y} = \frac{k_2(ic\omega + k_1)}{k_1k_2 + ic\omega(k_2 - m_2\omega^2) - (k_2m_1 + k_1m_2 + k_2m_2)\omega^2 + m_1m_2\omega^4}$$

$$H2_Y(\omega) = \frac{X_2}{Y} = \frac{(i2\frac{\omega}{\omega_1}\xi_1 + 1)}{1 + i2\frac{\omega_2}{\omega_1}\frac{\omega}{\omega_2}\xi_1\left[1 - \frac{\omega^2}{\omega_2^2}\right] - \left[1 + \frac{k_2}{k_1} + \frac{\omega_2^2}{\omega_1^2}\right]\frac{\omega^2}{\omega_2^2} + \frac{\omega_2^2}{\omega_1^2}\frac{\omega^4}{\omega_2^4}}$$

Two DOF with floor vibration transfer function

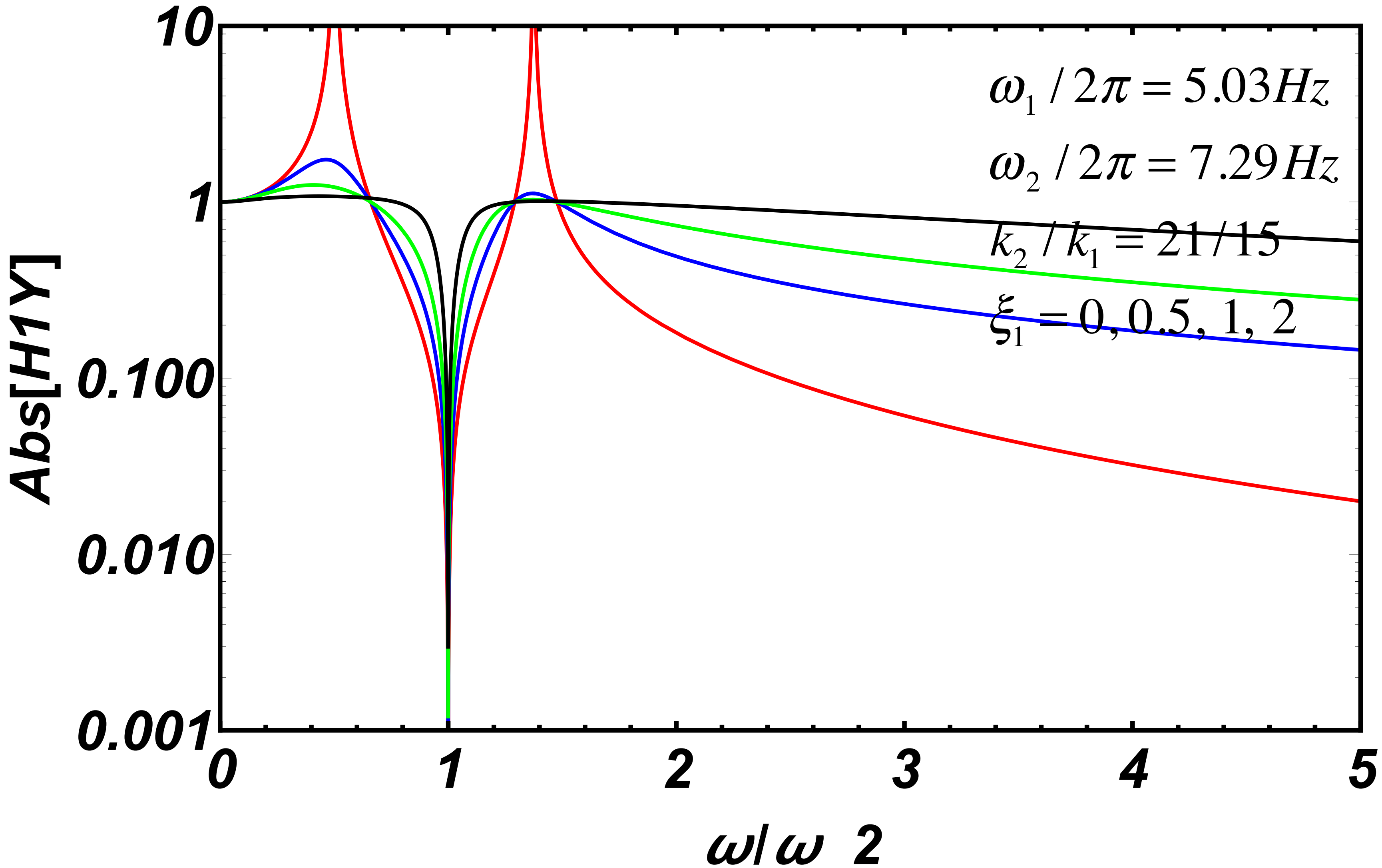
The frequencies chosen here are too high.



It is better to keep damping to low values. As damping is made stronger the resonant frequency shifts to ω_2 and the high frequency response goes back to single DOF.

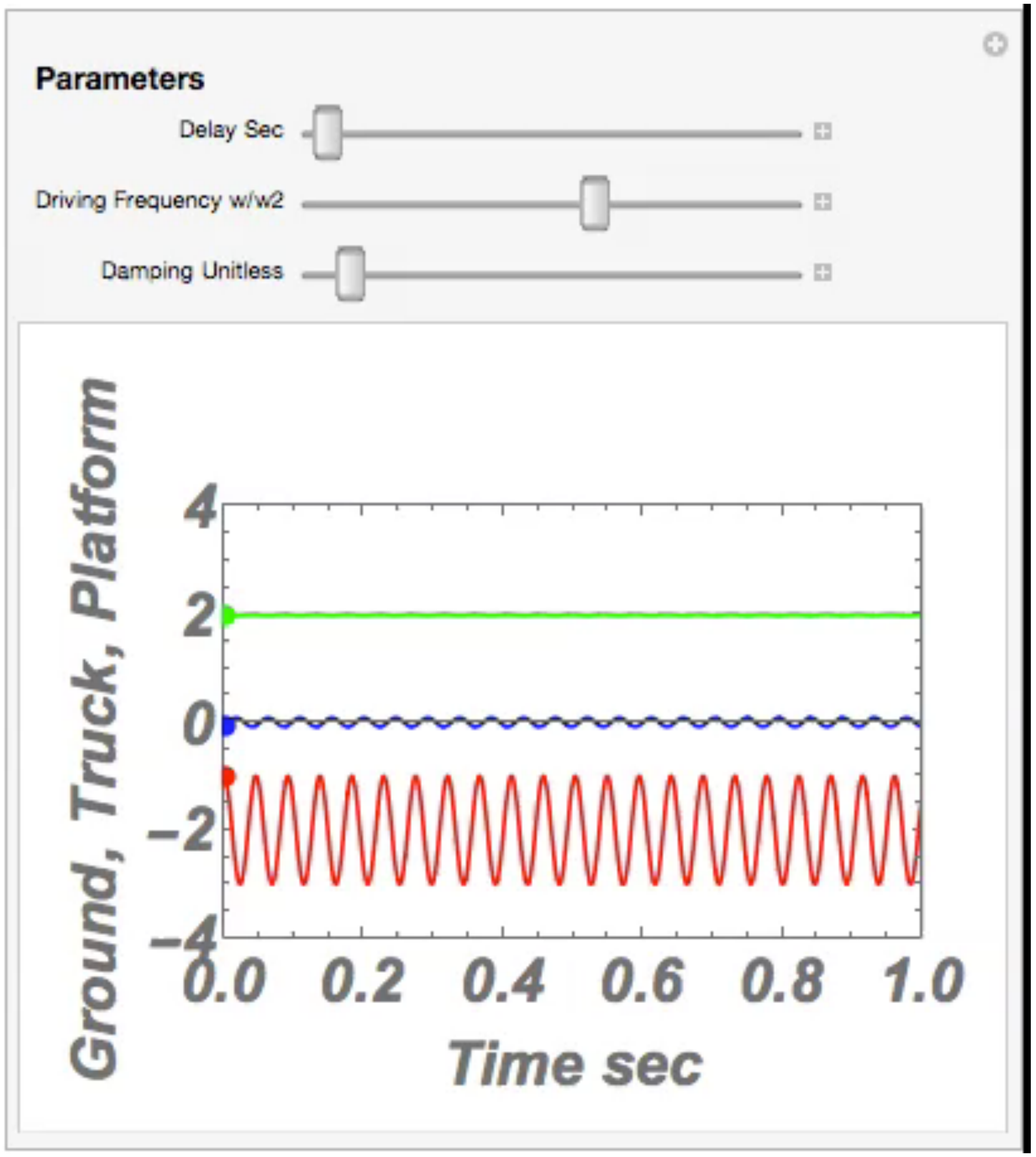
2 DOF transfer function for truck bed

$$H1_Y = \frac{X_1}{Y} = \frac{k_1}{k_2} \times \frac{-(1-\frac{\omega^2}{\omega_2^2})(i2\frac{\omega}{\omega_1}\xi_1 + 1)}{1-(1-\frac{\omega^2}{\omega_2^2})(1+\frac{k_1}{k_2}-\frac{m_1}{m_2}\frac{\omega^2}{\omega_2^2}+i2\frac{k_1}{k_2}\frac{\omega}{\omega_1}\xi_1)} \qquad \frac{\omega_2^2}{\omega_1^2} = \frac{k_2}{k_1}\frac{m_1}{m_2}$$

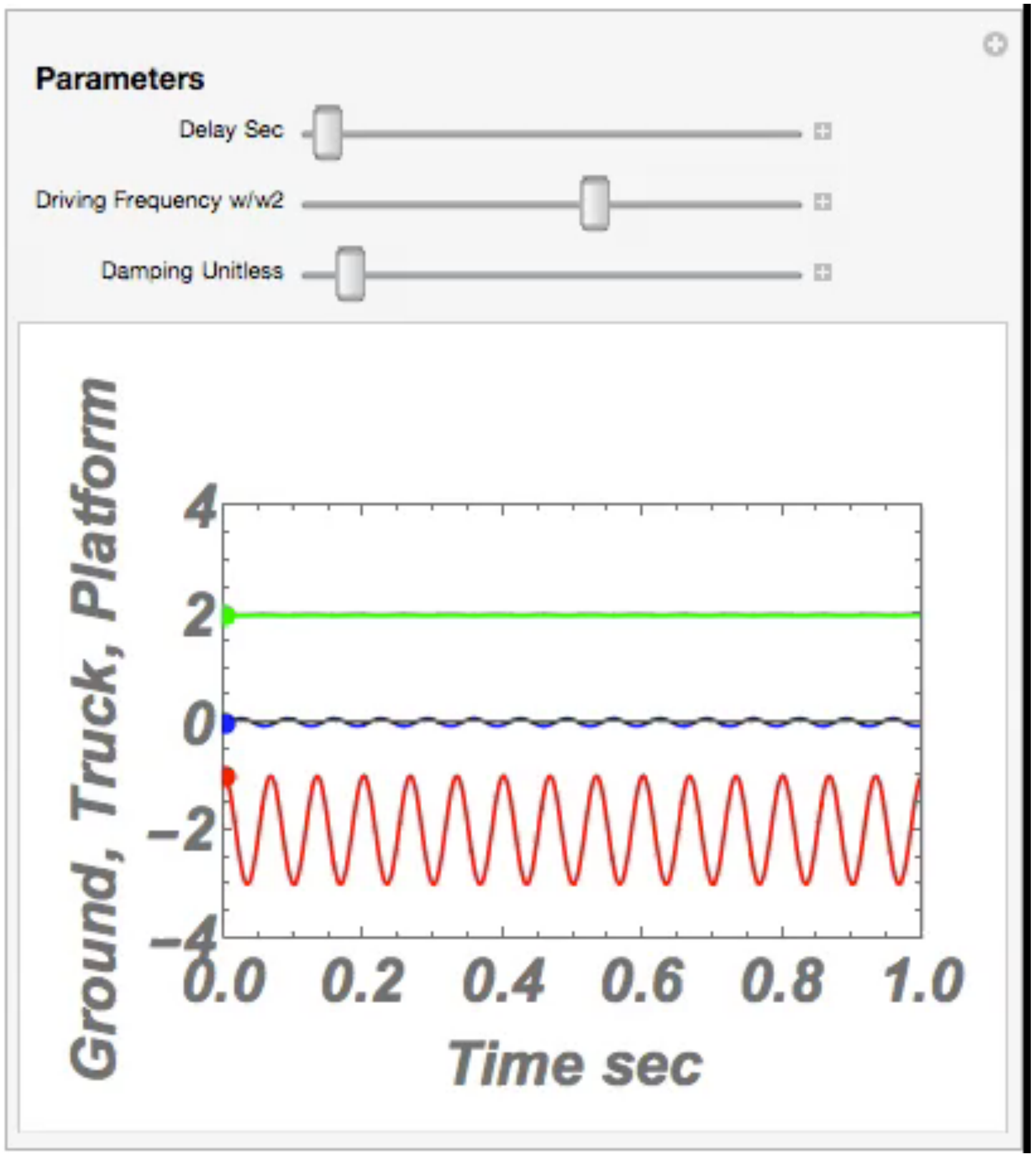


zero at $\omega/\omega_2=1$. At the this frequency the truck bed has no oscillation, the road vibration is in anti-correlation to the platform vibration.

Motion Simulation



$$\omega_1 / 2\pi = 5.03 Hz$$
$$\omega_2 / 2\pi = 7.29 Hz$$
$$m_2 / m_1 = 10Ton / 15ton$$



$$\omega_1 / 2\pi = 2 Hz$$
$$\omega_2 / 2\pi = 3 Hz$$
$$m_2 / m_1 = 10Ton / 15ton$$

Recall that lower natural frequency means larger motion under initial load. This can lead to impractical parameters.

Motion due to sudden step

Assume that $y(t) = u(t) = \begin{cases} 1 \; if \;\; t \geq 0 \\ 0 \; if \;\; t < 0 \end{cases}$

Then $Y(s) = L[y(t)] = 1/s$

$h1y(s)$ and $h2y(s)$ can be obtained from the Fourier transforms by: $i\omega \rightarrow s$

$x1(t) = L^{-1}[h1y(s) \bullet Y(s)]$

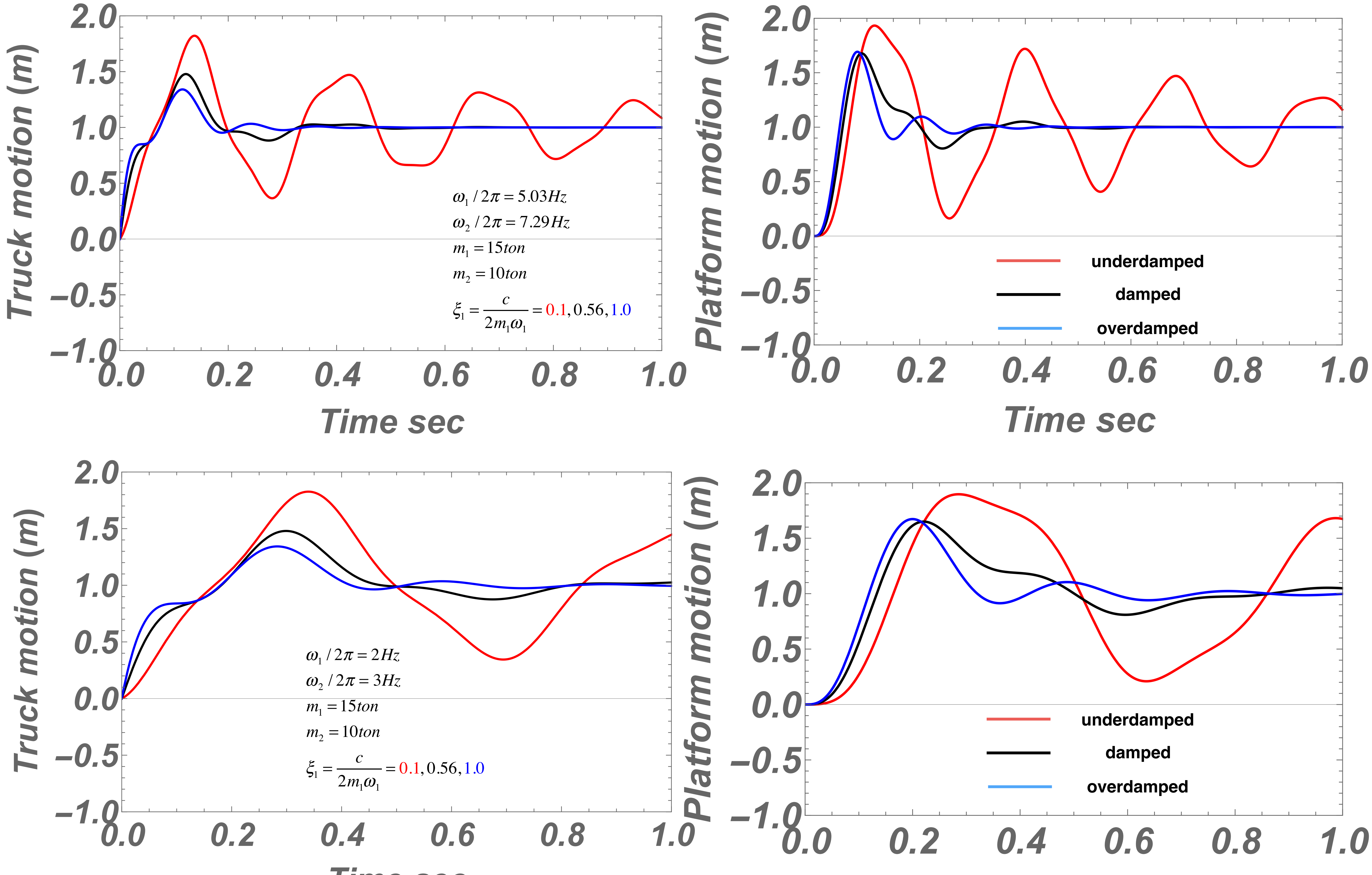
$x1''(t) = L^{-1}[s^2 \bullet h1y(s) \bullet Y(s)]$ ... Similarly for $x2(t)$, $x2''(t)$

For completeness...

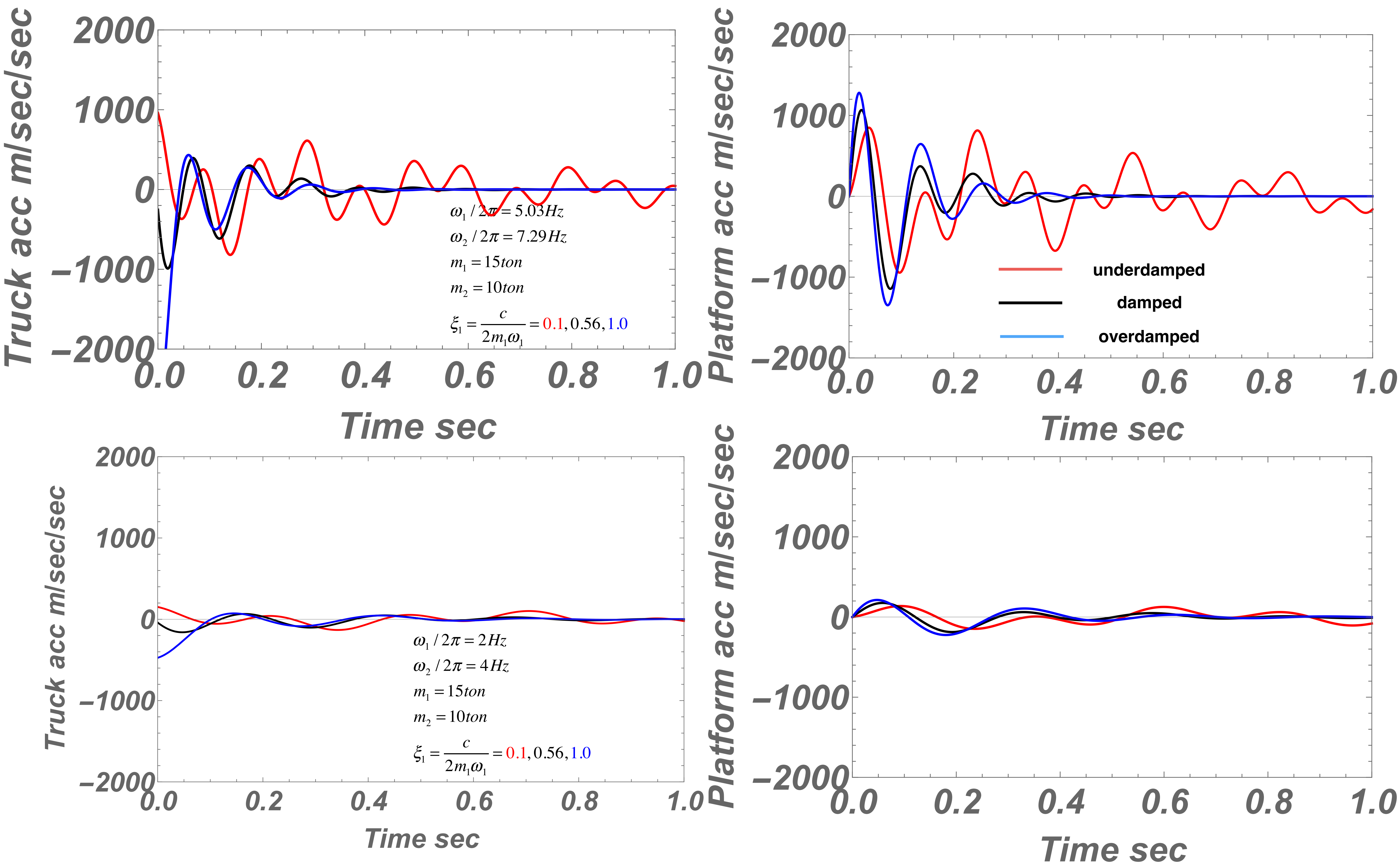
$$h1_Y(s) = \frac{k_1}{k_2} \times \frac{-(1+\frac{s^2}{\omega_2^2})(2\frac{s}{\omega_1}\xi_1 + 1)}{1-(1+\frac{s^2}{\omega_2^2})(1+\frac{k_1}{k_2}+\frac{m_1}{m_2}\frac{s^2}{\omega_2^2}+2\frac{k_1}{k_2}\frac{s}{\omega_1}\xi_1)}$$

$$h2_Y(s) = \frac{(2\frac{s}{\omega_1}\xi_1 + 1)}{1+2\frac{\omega_2}{\omega_1}\frac{s}{\omega_2}\xi_1\left[1+\frac{s^2}{\omega_2^2}\right]+\left[1+\frac{k_2}{k_1}+\frac{\omega_2^2}{\omega_1^2}\right]\frac{s^2}{\omega_2^2}+\frac{\omega_2^2}{\omega_1^2}\frac{s^4}{\omega_2^4}}$$

Motion of truck bed and platform after shock of 1 m vertical ground motion



Acceleration of truck bed and platform after shock of 1 m vertical ground motion



Conclusion

- ***Analysis of passive suspension was performed.***

- ***Rule of thumb for a single DOF suspension system: Factor of 3 detuning gets you factor 1/8 in response.***

- ***For 2 DOF further suppression of response is possible. (possibly another 1/10)***

- ***Damping actually worsens the suppression of the response.***

- ***Damping is needed to reduce response to shock. With optimum damping constant, motion can be damped within a single time of the lower oscillation frequency.***

- ***Sudden shock can result in high acceleration for brief periods of time. Higher damping causes higher acceleration. Softer spring constants can lower acceleration, but will cause large initial displacement.***

- ***In the 2 DOF system we showed how there are two normal modes and initial conditions determine how much of each mode contributed to the vibration. We will develop this subject more in a later lecture.***

- ***Please use the Mathematica notebook provided with this to play with the model. Unfortunately, I did not think about putting enough commentary in the notebook.***